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THE REYNOLDS ANALOGY APPLIED TO FLOW BETWEEN A ROTATING AND A STATIONARY DISC

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Abstract—This paper describes how Dorfman's [1] application of the Reynolds analogy for the free disc can be extended to include the effects of frictional dissipation in compressible or incompressible flow. For turbulent flow the 'effective' Prandtl number is taken to be unity, and the radial and tangential 'effective' viscosities are assumed to be equal. The adiabatic disc temperature is found to be identical with that obtained by previous authors [6, 7] for laminar flow and a unity Prandtl number, and is consistent with a flat plate adiabatic temperature. It is shown that under certain conditions the Reynolds analogy can be applied to a disc rotating near a stator, with and without a superimposed radial outflow of fluid. For the case of a forced outflow, the Reynolds analogy is applied to measured moment coefficients and shows that the Nusselt number is controlled by mass flow rate at high ratios of radial Reynolds number to ratational Reynolds number, and is governed primarily by rotational Reynolds number at low Reynolds number ratios.

NOMENCLATURE

c, constant of proportionality in disc temperature distribution;
C_m, moment coefficient for both sides of the rotating disc, $\equiv 2M/(\frac{1}{2}\rho\omega^2 r_0^5)$;
C_{m0}, moment coefficient for stator-side of the rotating disc, $\equiv M_0/(\frac{1}{2}\rho\omega^2 r_0^5)$;
C_p, specific heat at constant pressure;
C_w, mass flow coefficient, $\equiv W/(\mu r_0)$;
G, gap ratio, $\equiv s/r_0$;
G_c, shroud clearance ratio, $\equiv s_c/r_0$;
$\bar{h}$, total enthalpy;
M, M_0, moment on free side or stator-side of rotating disc, respectively;
n, exponent in disc temperature distribution;
Nu, local Nusselt number, $\equiv q_0 r/[\lambda(T_0 - T_{0,\mathrm{ad}})]$;
$\overline{Nu}$, average Nusselt number, $\equiv q_{0,\mathrm{av}} r/[\lambda(T_0 - T_{0,\mathrm{ad}})_{\mathrm{av}}]$;
p, static pressure;
Pr, Pr_t, laminar and turbulent Prandtl numbers, respectively;
q, heat flux;
r, r_0, arbitrary radius and disc radius, respectively;
Re, rotational Reynolds number, $\equiv \rho\omega r^2/\mu$;
Re_r, radial to rotational Reynolds number ratio, $\equiv C_w/(2\pi G Re)$;
s, s_c, axial clearance between rotor and stator and rotor and shroud, respectively;
T, absolute temperature;
V_r, V_ϕ, V_z, radial, tangential and axial velocity components respectively;
W, superimposed mass flow rate;
z, axial distance normal to rotor;
α, thermal diffusivity, $\equiv \lambda/(\rho C_p)$;
β, volume expansion coefficient, $\equiv -1/\rho(\partial\rho/\partial T)_p$;
ξ, radial coordinate;
Θ, dimensionless temperature, $\equiv (T - T_\infty)/(T_0 - T_\infty)$;
$\lambda, \lambda_{\mathrm{eff}}$, laminar and 'effective' thermal conductivity, respectively;
$\mu, \mu_{r,\mathrm{eff}}, \mu_{\phi,\mathrm{eff}}$, laminar, and 'effective' radial and tangential viscosities, respectively;
ν, kinematic viscosity;
ρ, density;
τ_r, τ_ϕ, radial and tangential shear stress components, respectively;

Φ, dimensionless velocity, $\equiv V_\phi/(\omega r)$;
ψ, stream function;
ω, angular velocity of the rotating disc.

Suffixes
ad, adiabatic;
av, radially-weighted average;
eff, effective, in turbulent flow;
o, pertaining to the rotor;
r, ϕ, z, radial, tangential and axial directions, respectively;
s, pertaining to the stator;
t, turbulent condition;
∞, pertaining to the free stream (free discs).

1. INTRODUCTION

THE REYNOLDS analogy has two roles in the field of heat-transfer: it provides an idealised theoretical model that can serve as a datum for more complicated mathematical models, and it can be readily modified to allow estimates of heat-transfer to be made from fluid dynamics data, which—in general—are easier to obtain than heat-transfer data. The rotating disc can serve as a simple model of a turbine rotor, and by using the analogy between heat and momentum it is possible to lay the foundation for heat-transfer predictions in the more complex turbine system. It is therefore instructive for the engineer to know under what circumstances the analogy can be used for rotating disc systems, and to have an estimate of the effect of such parameters as the clearance between the rotor and its housing, coolant mass flow rate, and rotational speed on the heat-transfer from a turbine rotor.

2. THE BOUNDARY LAYER EQUATIONS

For steady axi-symmetric flow over a rotating disc the continuity equation, the radial and tangential momentum equations and the energy equation can be written as:

$$\frac{\partial}{\partial r}(\rho r V_r) + \frac{\partial}{\partial z}(\rho r V_z) = 0 \qquad (1)$$

$$\rho\left(V_r \frac{\partial V_r}{\partial r} + V_z \frac{\partial V_r}{\partial z} - \frac{V_\phi^2}{r}\right) = -\frac{\mathrm{d}p}{\mathrm{d}r} + \frac{\partial \tau_r}{\partial z} \qquad (2)$$

$$\rho\left(V_r \frac{\partial V_\phi}{\partial r} + V_z \frac{\partial V_\phi}{\partial z} + \frac{V_\phi V_r}{r}\right) = \frac{\partial \tau_\phi}{\partial z} \qquad (3)$$

$$\rho C_p \left(V_r \frac{\partial T}{\partial r} + V_z \frac{\partial T}{\partial z}\right) = V_r T \beta \frac{\mathrm{d}p}{\mathrm{d}r} - \frac{\partial q}{\partial z} + \tau_r \frac{\partial V_r}{\partial z} + \tau_\phi \frac{\partial V_\phi}{\partial z}. \qquad (4)$$

For turbulent flow, the shear stresses and axial heat flux can be expressed as:

$$\tau_r = \mu \frac{\partial V_r}{\partial z} - \rho \overline{V_r' V_z'} \qquad (5)$$

$$\tau_\phi = \mu \frac{\partial V_\phi}{\partial z} - \rho \overline{V_\phi' V_z'} \qquad (6)$$

$$q = -\ \lambda \frac{\partial T}{\partial z} - \rho C_p \overline{T' V_z'}) \qquad (7)$$

where the primes denote the turbulent terms, and all values are time-averaged.

It will now be shown that a strong analogy exists between equations (3) and (4).

3. THE REYNOLDS ANALOGY APPLIED TO THE FREE DISC, NEGLECTING FRICTIONAL HEATING

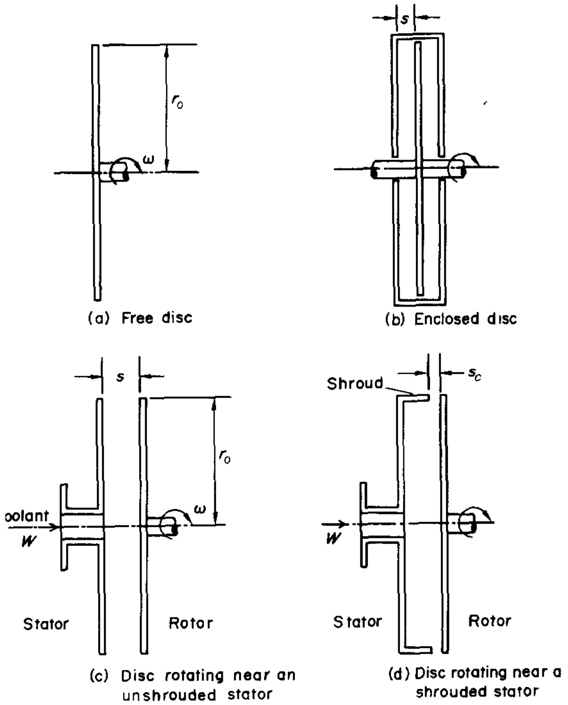
Dorfman [1] placed the Reynolds analogy on a firm mathematical basis for the free disc, shown in Fig. 1a, by showing that a strong similarity exists between the energy equation and the tangential momentum equation. This similarity can be best seen by defining a dimensionless temperature Θ, where $\Theta(r, z) \equiv (T - T_\infty)/(T_0 - T_\infty)$, T_0 and T_∞ being the disc temperature and the free stream temperature respectively, and a dimensionless velocity, Φ, where $\Phi(r, z) \equiv V_\phi/(\omega r)$.

It is further assumed that the free stream has zero velocity and is isothermal, and that the disc temperature is quadratic such that $T_0 = cr^2$, where c is constant. Under these conditions equations (3) and (4) can be rewritten for incompressible flow ($\beta = 0$), with zero dissipation, as

$$V_r \frac{\partial \Phi}{\partial r} + 2\Phi \frac{V_r}{r} + V_z \frac{\partial \Phi}{\partial z} = \frac{\partial}{\partial z}\left(\nu \frac{\partial \Phi}{\partial z} - \overline{\Phi' V_z'}\right) \quad (8)$$

$$V_r \frac{\partial \Theta}{\partial r} + 2\Theta \frac{V_r}{r} + V_z \frac{\partial \Theta}{\partial z} = \frac{\partial}{\partial z}\left(\alpha \frac{\partial \Theta}{\partial z} - \overline{\Theta' V_z'}\right) \quad (9)$$

As the boundary conditions are identical for both equations, such that $\Phi(r, 0) = \Theta(r, 0) = 1$



(a) Free disc (b) Enclosed disc

(c) Disc rotating near an unshrouded stator (d) Disc rotating near a shrouded stator

FIG. 1.

and $\Phi(r, \infty) = \Theta(r, \infty) = 0$, and if the Prandtl number is unity such that $\nu = \alpha$ the solutions of equations (8) and (9) will, for similar initial conditions, be identical so that $\Phi(r, z) = \Theta(r, z)$. This result, obtained by Dorfman, implicity assumes that the turbulent Prandtl number, Pr_t, is unity, where

$$Pr_t = \frac{\overline{V_\phi' V_z'} \Big/ \dfrac{\partial V_\phi}{\partial z}}{\overline{T' V_z'} \Big/ \dfrac{\partial T}{\partial z}}. \quad (10)$$

The variation of Pr_t across a boundary layer has been discussed by Kestin and Richardson [2], and it is a useful expedient to regard Pr_t as constant. As equations (8) and (9) are so strongly related, and are further constrained by their boundary conditions, the assumption that Pr_t should be unity does not seem unreasonable. Under these conditions it follows that the local heat flux can be determined by

$$q = -\frac{C_p \tau_\phi (T - T_\infty)}{\omega r}.$$

The local Nusselt number can be defined by $Nu \equiv q_0 r/[\lambda(T_0 - T_{0,\mathrm{ad}})]$, and the average Nusselt number by $\overline{Nu} \equiv q_{0,\mathrm{av}} r/[\lambda(T_0 - T_{0,\mathrm{ad}})_{\mathrm{av}}]$, where q_0 is the heat flux at the disc surface, and the sufiixes ad and av refer to the adiabatic and radially-weighted average values, respectively. As frictional heating has been neglected, the adiabatic disc temperature $T_{0,\mathrm{ad}}$ is obviously equal to T_∞. From these definitions it follows that:

$$Nu = -Re \frac{\tau_{\phi 0}}{\rho \omega^2 r^2} \quad (11)$$

and

$$\overline{Nu} = \frac{Re C_m}{2\pi} \quad (12)$$

where the rotational Reynolds number is defined as $Re \equiv \omega r^2/\nu$ and the moment coefficient as $C_m \equiv 2M/(\frac{1}{2}\rho\omega^2 r^5)$, where M is the frictional moment on one side of the disc. For local Nusselt numbers, r is the local radius, whilst for average values it is usual to take $r = r_0$, the disc radius.

Dorfman also made allowances for the influence of Prandtl number and the effect of other temperature profiles on the Nusselt number. Using Cochran's [3] moment coefficients for laminar flow and Karman's [4] for turbulent flow, equation (12) was shown to give good agreement with the experiments conducted by Cobb and Saunders [5] on an isothermal free disc rotating in air.

4. THE REYNOLDS ANALOGY APPLIED TO THE FREE DISC, INCLUDING FRICTIONAL HEATING

In order to extend the Reynolds analogy to compressible flow systems where frictional heating is significant, it is convenient to employ the von Mises transformation using the stream function, ψ, where

$$V_r = \frac{1}{\rho r}\frac{\partial \psi}{\partial z} \text{ and } V_z = -\frac{1}{\rho r}\frac{\partial \psi}{\partial r}.$$

Equations (2)–(4) can be written in ξ, ψ co-ordinates, where $\xi \equiv r$, such that:

$$\frac{\partial V_r}{\partial \xi} - \frac{V_\phi^2}{r V_r} = -\frac{1}{\rho V_r}\frac{\mathrm{d}p}{\mathrm{d}\xi} + r\frac{\partial \tau_r}{\partial \psi} \tag{13}$$

$$\frac{\partial V_\phi}{\partial \xi} + \frac{V_\phi}{r} = r\frac{\partial \tau_\phi}{\partial \psi} \tag{14}$$

$$C_p\frac{\partial T}{\partial \xi} = \frac{T\beta}{\rho}\frac{\mathrm{d}p}{\mathrm{d}\xi} + r\left(-\frac{\partial q}{\partial \psi} + \tau_r\frac{\partial V_r}{\partial \psi} + \tau_\phi\frac{\partial V_\phi}{\partial \psi}\right). \tag{15}$$

Using equations (13) and (14) to replace the dissipation terms in equation (15) it follows that:

$$C_p\frac{\partial T}{\partial \xi} = \frac{(T\beta - 1)}{\rho}\frac{\mathrm{d}p}{\mathrm{d}\xi} + r\frac{\partial}{\partial \psi}(-q + V_r\tau_r + V_\phi\tau_\phi) - \left(V_r\frac{\partial V_r}{\partial \xi} + V_\phi\frac{\partial V_\phi}{\partial \xi}\right). \tag{16}$$

It is now necessary to make some assumptions concerning the turbulent fluxes, and it is convenient to introduce the effective transport properties λ_{eff}, $\mu_{\phi,\text{eff}}$ and $\mu_{r,\text{eff}}$ where

$$\lambda_{\text{eff}} \equiv \left(\lambda\frac{\partial T}{\partial z} - \rho\overline{C_p T' V_z'}\right)\Big/\frac{\partial T}{\partial z} \tag{17}$$

$$\mu_{\phi,\text{eff}} \equiv \left(\mu\frac{\partial V_\phi}{\partial z} - \rho\overline{V_\phi' V_z'}\right)\Big/\frac{\partial V_\phi}{\partial z} \tag{18}$$

$$\mu_{r,\text{eff}} \equiv \left(\mu\frac{\partial V_r}{\partial z} - \rho\overline{V_r' V_z'}\right)\Big/\frac{\partial V_r}{\partial z}. \tag{19}$$

For the case of a unity Prandtl number it will be assumed, as in Section 3, that Pr_t is also unity, which implies that:

$$\lambda_{\text{eff}} = C_p\mu_{\phi,\text{eff}}. \tag{20}$$

In the calculation of moment coefficients for the free disc [1, 4] it is assumed that the radial and tangential shear stress on the rotating disc are in the ratio of the radial and tangential velocity components near the surface. This implicity assumes that $\mu_{r,\text{eff}} = \mu_{\phi,\text{eff}}$, and so for convenience these effective viscosities will be assumed equal to a common value, μ_{eff}, and equation (16) can be rewritten for a unity Prandtl number (where $\mu_{\text{eff}} = \lambda_{\text{eff}}/C_p$), as

$$\frac{\partial}{\partial \xi}(C_p T + \tfrac{1}{2}V_r^2 + \tfrac{1}{2}V_\phi^2) + \left(\frac{1 - T\beta}{\rho}\right)\frac{\mathrm{d}p}{\mathrm{d}\xi} = r\frac{\partial}{\partial \psi}\left[\rho r V_r \mu_{\text{eff}}\frac{\partial}{\partial \psi}(C_p T + \tfrac{1}{2}V_r^2 + \tfrac{1}{2}V_\phi^2)\right]. \tag{21}$$

Equation (21) can be considerably simplified for the case of an incompressible fluid, where $\beta = 0$, and for a perfect gas, where $\beta = 1/T$. For each of these cases it is convenient to introduce a total enthalpy, $\bar{h}$, where:

$\bar{h} = C_p T + \frac{1}{2}V_r^2 + \frac{1}{2}V_\phi^2$: perfect gas

$\bar{h} = C_p T + \frac{1}{2}V_r^2 + \frac{1}{2}V_\phi^2 + p/\rho$: incompressible fluid

and equation (21) can be rewritten as:

$$\frac{\partial \bar{h}}{\partial \xi} = r\frac{\partial}{\partial \psi}\left[\rho r V_r \mu_{\text{eff}}\frac{\partial \bar{h}}{\partial \psi}\right]. \tag{22}$$

In order to show the similarity between this energy equation and the tangential momentum equation, equation (14) can be expressed as:

$$\frac{\partial}{\partial \xi}(r V_\phi) = r\frac{\partial}{\partial \psi}\left[\rho r V_r \mu_{\text{eff}}\frac{\partial}{\partial \psi}(r V_\phi)\right]. \tag{23}$$

The analogy between equations (22) and (23) is complete if the initial and boundary conditions are similar, such that:

$$\psi = \psi_0 : rV_{\phi 0} = \omega r^2, \quad \bar{h}_0 = \mathrm{c}r^2$$

$$\psi = \psi_\infty : rV_{\phi\infty} = 0, \quad \bar{h}_\infty = \text{constant}.$$

Under these conditions, the distribution of tangential velocity and total enthalpy will be similar, and as a consequence:

$$\frac{\bar{h} - \bar{h}_\infty}{\bar{h}_0 - \bar{h}_\infty} = \frac{V_\phi}{\omega r}. \tag{24}$$

In particular

$$q = -\mu_{\text{eff}} \frac{\partial}{\partial z}(C_p T)$$

$$= -\mu_{\text{eff}} \frac{\partial \bar{h}}{\partial z} + V_r \tau_r + V_\phi \tau_\phi$$

whence from equation (24)

$$q = V_r \tau_r + V_\phi \tau_\phi - \frac{\bar{h}_0 - \bar{h}_\infty}{\omega r} \tau_\phi. \tag{25}$$

On the rotating disc the heat flux is given by:

$$q_0 = -\frac{\tau_{\phi,0}}{\omega r}[C_p(T_0 - T_\infty) - \tfrac{1}{2}\omega^2 r^2] \tag{26}$$

which leads to the important result that for the case of $q_0 = 0$ the adiabatic disc temperature, for an incompressible fluid or a perfect gas, $T_{0,\text{ad}}$, is given by:

$$T_{0,\text{ad}} = T_\infty + \tfrac{1}{2}\omega^2 r^2 / C_p. \tag{27}$$

This result agrees with the findings of Riley [6] and Mabuchi *et al.* [7], who solved the incompressible laminar energy equation for the free disc. It is interesting to observe that $T_{0,\text{ad}}$ is unaffected by the radial velocity, and corresponds to an adiabatic plate temperature for a free stream velocity of ωr.

As a consequence of equation (26) it is apparent that the Nusselt numbers given by equations (11) and (12) are valid, within the limits of the assumptions made, even when frictional heating is significant, providing equation (27) is employed for the adiabatic disc temperature used in defining the Nusselt numbers.

5. THE REYNOLDS ANALOGY APPLIED TO A DISC ROTATING NEAR A STATOR

The turbulent free disc analysis of Karman has been extended by Schultz-Grunow [8] and Daily and Nece [9] to produce moment coefficients for the enclosed disc, illustrated in Fig. 1b. Also Bayley and Owen [10] and

Table 1. Moment coefficients and Nusselt numbers for forced radial outflow between a rotating and a stationary disc

Author	Measurements	Empirical correlation	Range
Sedach [11]	Moment coefficients	$C_{M0} = 0{\cdot}078\, Re^{-0\cdot 2} + 0{\cdot}127\,(C_W/GRe)^{0\cdot 5}\, G$	$0{\cdot}1 < G < 0{\cdot}3$ $0{\cdot}25 \times 10^6 < Re < 1{\cdot}7 \times 10^6$ $0 < Re_r < 0{\cdot}059$
Kreith *et al.* [12]	Mass transfer ($Pr = 2{\cdot}4$)	$\bar{N}u = (\tfrac{1}{2}G)^{0\cdot 55}[1{\cdot}36 + 1{\cdot}29(Re/10^5) + 3{\cdot}57(Re/10^5)^2 - 3{\cdot}51(Re/10^5)^3 + 1{\cdot}84(Re/10^5)^4](C_w/2\pi G)^{(0\cdot 83 - 0\cdot 12\, Re/10^5)}$	$0{\cdot}012 < G < 0{\cdot}06$ $0 < Re < 4 \times 10^4$ $0{\cdot}13 < Re_r < \infty$
Mitchell [13]	Heat transfer ($Pr = 0{\cdot}72$)	$\bar{N}u = 1{\cdot}22\, C_W^{0\cdot 126}$	$G = 0{\cdot}113$ $1{\cdot}7 \times 10^4 < C_W < 7{\cdot}4 \times 10^4$ $Re_r = 0{\cdot}53, 0{\cdot}29, 0{\cdot}14$
Kapinos [14]	Heat transfer ($Pr = 0{\cdot}72$)	$\bar{N}u = 0{\cdot}0261\,(C_W/GRe)^{0\cdot 1}\, Re^{0\cdot 8}\, G^{0\cdot 06}$	$0{\cdot}016 < G < 0{\cdot}06$ $0{\cdot}5 \times 10^6 < Re < 4 \times 10^6$ $0{\cdot}012 < Re_r < 0{\cdot}1$

Sedach [11] have measured moment coefficients for the case of a disc and stator with a radial outflow of coolant, as illustrated in Fig. 1c. For the latter case, it has been shown that for small gap ratios ($G < 0{\cdot}1$, where $G \equiv s/r_0$) the whole space between the rotor and stator can be treated as a boundary layer. Whilst it is not necessarily true that $\mu_{r,\,\mathrm{eff}} = \mu_{\phi,\,\mathrm{eff}}$ for the case of a radial outflow of fluid, in practical cases—such as air-cooled turbine rotors—the tangential velocity component is usually larger than the radial component. Near the turbine rotor the dissipation due to the radial components is much smaller than that due to the tangential component, hence the radial dissipation terms have very little effect on heat transfer. Owen [15] has shown that equation (27) is still valid if the radial dissipation terms are neglected, but for convenience it will be considered that $\mu_{r,\,\mathrm{eff}} = \mu_{\phi,\,\mathrm{eff}}$ such that the results of Section 4 will still be applicable. As the frictional moment of the front—or stator side face—of the rotating disc will not, for radial outflow, be equal to that on the back—or free side—it is apparent that equation (12) will only be valid if it is modified to:

$$\overline{Nu} = Re\, C_{m,\,0}/\pi \qquad (28)$$

where $C_{m,\,0} \equiv M_0/(\tfrac{1}{2}\rho\omega^2 r_0^5)$, M_0 being the frictional moment on the stator-side face of the disc.

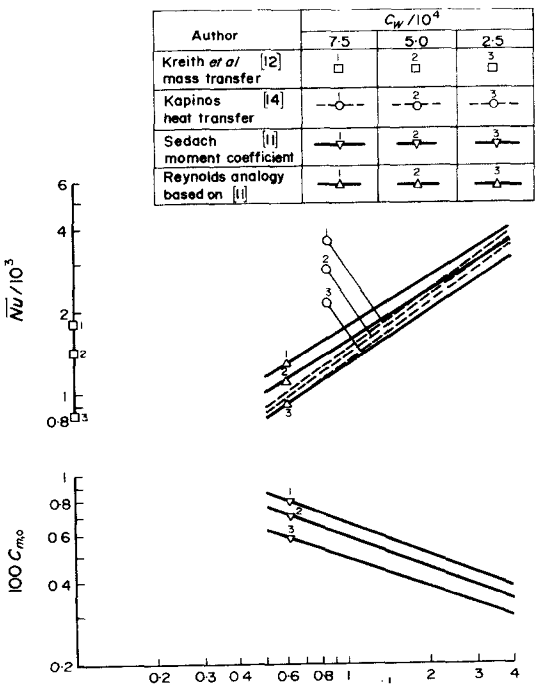


FIG. 2. Reynolds analogy applied to the moment coefficients of Sedach: $G = 0{\cdot}06$.

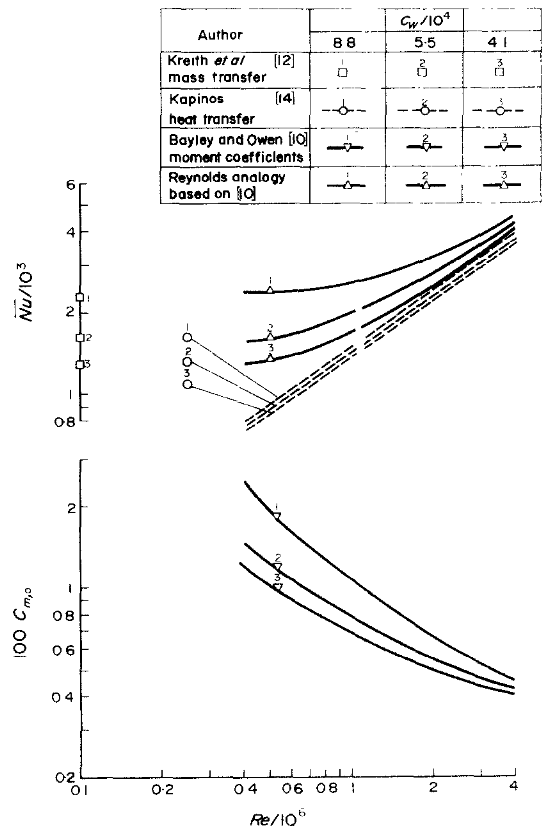


FIG. 3. Reynolds analogy applied to the moment coefficients of Bayley and Owen: $G = 0{\cdot}03$.

Owing to the assumptions made, equation (28) will only apply for $Pr = 1$ and for the boundary conditions

$$z = 0: V_{\phi,\,0} = \omega r,\ \bar{h}_0 = cr^2$$

$$z = s: V_{\phi,\,s} = 0,\ \ \bar{h}_s = \text{constant}.$$

However, in [15] it has been shown that Dorfman's corrections for the effect of disc temperature distribution and arbitrary Prandtl numbers on the Nusselt number are reasonably valid for the case of a radical outflow. These corrections can be summarised for turbulent flow as:

$$\bar{N}u(Pr) \doteqdot Pr^{0\cdot6}\,\bar{N}u(Pr = 1) \qquad (29)$$

$$\bar{N}u(n) \doteqdot \left(\frac{n + 2\cdot6}{4\cdot6}\right)^{0\cdot2} \bar{N}u(n = 2) \qquad (30)$$

where it is assumed that $\bar{h}_0 = cr^n$, c and n being constants.

The experiments of Kreith *et al.* [12], Mitchell [13] and Kapinos [14], which are summarised in Table 1, provide data for testing the Reynolds analogy applied to rotating discs with radial outflow. The results of Kreith *et al.* and Mitchell show little effect of rotational Reynolds number on the Nusselt number, but indicate that heat-transfer is governed primarily by mass flow rate. On the other hand, Kapinos'

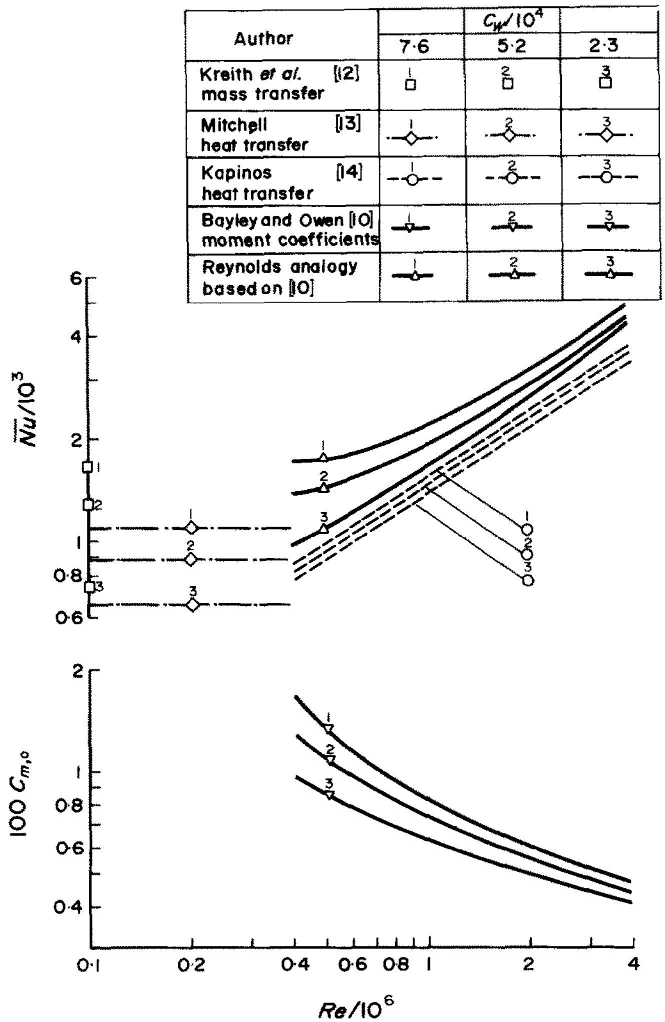


FIG. 4. Reynolds analogy applied to the moment coefficients of Bayley and Owen: $G = 0{\cdot}12$.

results show a marked effect of rotational Reynolds number on heat-transfer, with only a relatively small increase due to mass flow rate. For the Nusselt number to be independent of Reynolds number it is necessary—according to equation (28)—that $C_{m,0} \alpha Re^{-1}$. In a recent, as yet unpublished, analysis [18] using the integral momentum equations, solutions have been obtained showing that for $Re_r > 0{\cdot}875$, $C_{m,0} \alpha Re^{-1}$ [where $Re_r \equiv C_w/(2\pi G Re)$ and $C_w \equiv W/(\mu r_0)$]. The value of $Re_r > 0{\cdot}875$ stems from mathematical rather than physical arguments, but it is interesting to note from Table 1 that the values of Re_r for the experiments of Mitchell and Kreith *et al.* are larger than those of Sedach and Kapinos. Hence, it is not surprising that the Reynolds analogy applied to the results of Sedach in Fig. 2 show similar trends to the results of Kapinos. It should be pointed out that in Fig. 2, and subsequent figures, the effect of Prandtl number has been accounted for using equation (29). Also, as the experimental ranges of the various experiments do not in general coincide, it has been necessary to extrapolate results outside of their stated range of validity.

Figures 3 and 4 show the Reynolds analogy applied to the moment coefficients of Bayley and Owen, and a definite transition between the results of Kapinos and Kreith *et al.* can be seen. Mitchell's results, strictly valid for $G = 0{\cdot}113$, are shown in Fig. 3 to lie below the results of Kreith *et al.* Despite the fact that the experimental ranges do not coincide, the Reynolds analogy points out the transition between the rotation dominated regime and the area controlled by mass flow rate.

For the case of axial flow turbines, the configuration shown in Fig. 1d is a model of an air-cooled turbine rotor where peripheral shrouds are used to control the egress of coolant and to prevent the ingress of hot gases. The fluid dynamics of this system have been investigated [17], and whilst the presence of a shroud invalidates the boundary layer assumptions, a qualitative guide to heat-transfer should be gained by applying the Reynolds analogy to measured moment coefficients. The expected trends are shown in Fig. 5, where the effect of decreasing the clearance ratio G_c (where $G_c \equiv s_c/r_0$) is to increase the moment coefficients and hence the Nusselt numbers. It should be pointed out that the advantage of increased Nusselt number at small clearance ratios is likely to be offset by the increase in frictional

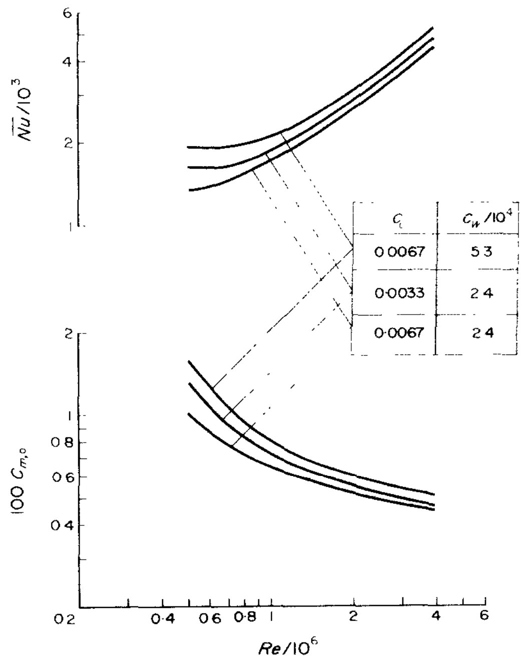


FIG. 5. Reynolds analogy applied to shrouded disc: $G = 0{\cdot}12$.

heating due to the presence of the shroud, and the adiabatic disc temperature given by equation (27) will not be valid.

6. CONCLUSIONS

This paper has been described the application of the Reynolds analogy to rotating disc systems, and the principal conclusions to be drawn from this work are:

(1) The Reynolds analogy can be applied to

the free disc when frictional heating is significant if

(i) the laminar and turbulent Prandtl numbers are unity;
(ii) the disc temperature varies quadratically with radius and the total enthalpy of the free stream is constant;
(iii) the 'effective' radial and tangential viscosities are equal;
(iv) the initial tangential velocity and total enthalpy distributions are similar.

Under these conditions, the average Nusselt number, $\bar{N}u$, is related to the moment coefficient, C_m, by $\bar{N}u = ReC_m/(2\pi)$; and for compressible or incompressible fluids the adiabatic disc temperature is related to the free stream temperature by: $T_{0,\mathrm{ad}} = T_\infty + \frac{1}{2}\omega^2 r^2/C_p$.

(2) The Reynolds analogy can be applied to the totally enclosed rotating disc, or the disc rotating near a stator with a radial outflow of coolant, if conditions (i), (ii), (iii) and (iv) apply—with the exception that the total enthalpy on the stator surface is constant. For the case of a forced outflow between a rotating and stationary disc, the mean Nusselt number is related to the moment coefficient of the stator-side rotor face, C_{m0}, by $\bar{N}u = ReC_{m0}/\pi$, and the adiabatic disc temperature is related to the stator temperature by $T_{0,\mathrm{ad}} = T_s + \frac{1}{2}\omega^2 r^2/C_p$.

(3) The Reynolds analogy applied to measured moment coefficients for the case of a forced radial outflow between a rotating and a stationary disc shows that the Nusselt number is controlled principally by mass flow rate when the ratio of radial to rotational Reynolds number, Re_r, exceeds a certain value (approximately, $Re_r > 0{\cdot}875$) and by rotational Reynolds number for Re_r less than the critical value. Nusselt numbers calculated in this way form a transition between heat transfer measurements at low Re_r and the measurements at high Re_r. For the case of a shrouded disc, the Nusselt number is expected to increase with decreasing shroud clearance—for a given mass flow rate—but the advantage of larger Nusselt numbers will be partially offset by an increase in frictional heating and higher adiabatic disc temperatures.

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L'ANALOGIE DE REYNOLDS APPLIQUÈE À UN ÈCOULEMENT ENTRE UN DISQUE TOURNANT ET UN DISQUE FIXE

Résumé—Cet article montre comment on peut élargir le principe de l'analogie de Reynolds appliqué par Dorfman [1] à un disque libre, de façon à tenir compte des effets de dissipation par frottement dans un écoulement compressible ou incompressible. Pour un écoulement turbulent le nombre "effectif" de Prandtl est pris égal à l'unité, et on suppose égales les viscosités radiales et tangentielles "effectives". La température adiabatique du disque trouvée correspond à celle obtenue antérieurement par les auteurs [6, 7] pour un écoulement laminaire et un nombre de Prandtl unitaire, et est en accord avec la température adiabatique d'une plaque plane. On montre que, sous certaines conditions, il est possible d'appliquer l'analogie de Reynolds à un disque tournant près d'un stator, avec ou sans écoulement radial surimposé du fluide. Dans le cas d'un écoulement forcé, l'analogie de Reynolds appliquée à des coefficients mesurés montre que le nombre de Nusselt est contrôlé par le débit massique pour des rapports élevés du nombre de Reynolds radial au nombre de Reynolds rotationnel, et qu'elle dépend en premier du nombre de Reynolds rotationnel pour des faibles rapports des nombres de Reynolds.

DIE REYNOLDS-ANALOGIE, ANGEWANDT AUF DIE STRÖMUNG ZWISCHEN EINER ROTIERENDEN UND EINER FESTSTEHENDEN SCHEIBE

Zusammenfassung—Diese Abhandlung beschreibt die Anwendung der Reynolds-Analogie nach Dorfmann (1) für eine freie Scheibe, um Reibungseffekte in kompressibler und inkompressibler Strömung einzubeziehen. In turbulenter Strömung wird die effektive Prandtl-zahl gleich eins gesetzt. Die effektiven Zähigkeiten in radialer und tangentialer Richtung werden einander gleichgesetzt. Die adiabate Scheibentemperatur erweist sich als identisch mit den Ergebnissen früherer Autoren (6) (7) für laminare Strömung und einer Prandtl-zahl gleich eins und als übereinstimmend mit der adiabaten Temperatur einer ebenen Platte. Es wird gezeigt, dass unter gewissen Bedingungen die Reynoldanalogie auf eine in der Nähe des Stators rotierende Scheibe angewandt werden kann, mit und ohne einer radial nach aussen überlagerten Strömung des Fluids. Für den Fall einer erzwungenen Strömung nach aussen wird die Reynoldsanalogie auf die gemessenen momentanen Koeffizienten angewandt. Es zeigt sich, dass die Nusselt-Zahl bei grossen Verhältnissen von radialer Reynoldszahl zur Reynoldszahl der Rotation vom Massenstrom und bei niedrigen Reynoldzahlen vorwiegend von der Reynoldszahl der Rotation bestimmt wird.

ПРИМЕНЕНИЕ АНАЛОГИИ РЕЙНОЛЬДСА К ТЕЧЕНИЮ МЕЖДУ ВРАЩАЮЩИМСЯ И НЕПОДВИЖНЫМ ДИСКОМ

Аннотация—Показано, как применение аналогии Рейнольдса к свободному диску, предложенное Дорфманом (I), может быть расширено, включая влияние диссипации трением в сжимаемом или несжимаемом потоке. Сделано допущение, что для турбулентного потока «эффективное» число Прандтля равно единице, а радиальная и тангенциальная «эффективная» вязкость равны. Найдено, что адиабатическая температура диска тождественна температуре, полученной авторами ранее (6), (7) для ламинарного потока при числе Прандтля I, и согласуется с адиабатической температурой плоской пластины. Показано, что при определенных условиях аналогия Рейнольдса может быть применена к диску, вращающемуся возле неподвижного, с наложенным радиальным оттоком жидкости и без оттока. Для случая вынужденного оттока аналогия Рейнольдса применена к измеренным мгновенным коэффициентам, и показано, что число Нуссельта зависит от массовой скорости при больших отношениях радиального числа Рейнольдса к вращательному числу Рейнольдса и определяется, главным образом, вращательным числом Рейнольдса при малых отношениях чисел Рейнольдса.